Domination Number of the Rough Zero Divisor Graph of the Rough Semiring

B. Praba, M. Logeshwari, A. Manimaran

Received 05 November 2018 • Revised: 23 November 2018 • Accepted: 02 December 2018

Abstract: In this paper, we consider an approximation space \( I = (U, R) \) where \( U \) is non-empty finite set and \( R \) is an arbitrary equivalence relation on \( U \). We define the dominating set of the Rough zero divisor graph \( G(Z(T^*) \) of the Rough Semiring \( T \), where \( Z(T^*) \) denotes the set of nonzero zero divisor of \( T \). We construct a minimal dominating set on \( T \). We also prove that the domination number of \( G(Z(T^*)) \) is equal to the number of equivalence classes induced by \( R \) on \( U \). We illustrate these concepts with suitable examples.

Keywords: Domination number, Dominating set, Rough Semiring, Semiring, Zero divisor, Zero divisor graph.

INTRODUCTION

Z. Pawlak [10] introduced the concept of rough set in 1982, a formal tool to process the incomplete information in the information system and it is defined as a pair of sets called lower and upper approximation. Hong et al. [7] dealt with some resultants over commutative idempotent Semiring in 2017. Zadeh [16] introduced the concept of fuzzy set theory in 1965 which is completely new approach to deal with vagueness and it is defined by membership in contrast to crisp membership. In 2013, Praba et al. [11] [12] [13] [14] gave a lattice structure on the set of all Rough sets on a given information system \( I = (U, A) \). The least upper bound and greatest lower bound were established using the operations \( \Praba \Delta \) and \( \Praba \nabla \). The author proved that \( T \) is not a Boolean algebra and derived the necessary condition for the existence of the complements of \( T \). Hence \( (T, \Delta, \nabla) \) is a Semiring called a Rough Semiring. In [8] [9] the authors detailed study of the rough semiring and its homomorphism were established. S. Akbari et al. [1] [2] discussed The Zero Divisor Graph of a Commutative Ring. David Dolzan and Polana Oblak [3] studied the concept of The Zero Divisor Graphs of Rings and Semirings.

In this paper, in section 2 we give preliminary definition that are required to study the forthcoming sections. In section 3, we prove that the cardinality of the minimal dominating set is equal to the number of equivalence classes induced by \( R \) on \( U \) and finally, we illustrate these concepts through examples in section 4.

Preliminaries

In this section we give preliminary definitions related to rough set theory and graph theory.

Rough Set Theory

Let \( U \) be a non empty finite set and \( R \) be an arbitrary equivalence relation on \( U \) then \( I = (U, R) \) is called an approximation space and for \( x \in U \), \( [x]_R = \{ y \in U \mid (x, y) \in R \} \) is said to be an equivalence class. Then, for \( X \subseteq U \), let \( RS(X) = \{ R(X), \overline{R}(X) \} \) be the rough set, where \( R(X) = \{ x \in U \mid [x]_R \subseteq X \} \) is said to be a lower approximation and the upper approximation is defined as \( \overline{R}(X) = \{ x \in U \mid [x]_R \cap X \neq \emptyset \} \). Also the set of rough sets is defined as \( T = \{ RS(X) \mid X \subseteq U \} \).

Definition 2.1.1[13]

Let \( X, Y \subseteq U \). The Praba \( \Delta \) is defined as

\[ X \Delta Y = \{ x \in U \mid (x \in X \land y \notin X) \lor (x \notin X \land y \in X) \} \]

B. Praba, SSN College of Engineering, Kalavakkam, Chennai, Tamilnadu. E-mail: prabab@ssn.edu.in
M. Logeshwari, SSN College of Engineering, Kalavakkam, Chennai, Tamilnadu. E-mail: logeshjimail@gmail.com
A. Manimaran, VIT, Vellore, Tamilnadu. E-mail: marans2011@gmail.com
XΔY = X ∪ Y, if IW(X ∪ Y) = IW(X) + IW(Y) − IW(X ∩ Y).

If IW(X ∪ Y) > IW(X) + IW(Y) − IW(X ∩ Y), then identify the equivalence class obtained by the union of X and Y. Then delete the elements of that class belonging to Y. Call the new set as Y. Now, obtain XΔY. Repeat this process until IW(X ∪ Y) = IW(X) + IW(Y) − IW(X ∩ Y). Where IW(X) is the number of equivalence classes contained in X.

**Definition 2.1.2[13]**

If X, Y ⊆ U then an element x ∈ U is called a Pivot element, if [x]p /∈ X ∩ Y, but [x]p ∩ X ≠ ∅ and [x]p ∩ Y ≠ ∅. The set of Pivot elements of X and Y is called the Pivot set of X and Y and it is denoted by P_{X∩Y}

**Definition 2.1.3[13]**

Praba ∇ of X and Y is denoted by X∇Y and it is defined as X∇Y = {x | [x]p ⊆ X ∩ Y} ∪ PX∩Y where X, Y ⊆ U. Note that each Pivot element in P_{X∩Y} is the representative of that particular class.

**Theorem 2.1.1[13]**

Let I = (U, A) be an information system where U be the universal (finite) set and A be the set of attributes and T be the set of all rough sets then (T, ∆, ∇) is a Semiring.

**Definition 2.1.4[14]**

A subset X of U is said to be dominant if X ∩ X_i ≠ ∅ for i = 1,2,...,n.

**Definition 2.1.5[14]** (Rough zero divisor)

Let (T, ∆, ∇) be a commutative rough semiring. An element RS(X) ≠ RS(∅) of T is said to be a zero divisor of T if there exist RS(Y) ≠ RS(∅) in T such that RS(X)∇RS(Y) = RS(∅) i.e., RS( X∇Y) = RS(∅)

**Graph Theory**

**Definition 2.2.1[4]**

A set D of vertices in a graph G(V,E) is said to be dominating set of G if every vertex not in D is adjacent to at least one vertex in D.

**Definition 2.2.2[4]**

A dominating set S is a minimal dominating set if no proper subset of S is a dominating set. The cardinality of smallest minimal dominating set of G is called the domination number and it is denoted by γ(G).

**DOMINATION NUMBER OF ROUGH ZERO DIVISOR GRAPH**

We know that a set P ⊆ Z(T∗) is said to be a dominating set of the rough zero divisor graph G(Z(T∗)) if its satisfies the domination property namely if every element of Z(T∗) − P is adjacent to at least one element in P and P is said to be a minimal dominating set if the removal of any element in P will affect the domination property. The number of elements in the smallest minimal dominating set is called the domination number of G(Z(T∗)).

In this section we consider an approximation space I = (U, R) and let {X_1,X_2,...,X_n} are the equivalence classes induced by Ron U.

We also assume that there are m equivalence classes {X_1,X_2,...,X_m} with cardinality greater than 1 and the remaining n − m equivalence classes {X_{m+1},X_{m+2},...,X_n} have cardinality equal to 1, where m ≤ n. Let {x_1,x_2,...,x_m} are the pivot elements (representative elements) of the equivalence classes with cardinality greater than 1.

Let T = {RS(X)|X ⊆ U} be the Semiring with respect to the two operation prabaΔ and praba∇. Let D = {RS(x_1),RS(x_2),...RS(x_m),RS(X_{m+1}),RS(X_{m+2})...RS(X_n)}.

Also we prove that D is the smallest minimal dominating set thereby proving that the domination number of the rough zero divisor graph of the rough semiring is equal to the number of equivalence classes induced by R on U.

Let B_i = {x_i,X_i,∅} for i = 1,2,...n (Note that B_{m+1} = {X_{m+1},∅},B_{m+2} = {X_{m+2},∅}...B_n = {X_n,∅})

**Theorem**

The set D is minimal dominating set for the rough zero divisor graph G(Z(T∗)) of the rough Semiring T.
Proof

First let us prove that \( D \) is the dominating set. Choose an element \( RS(Y) \in Z(T^*) \). If \( Y \) contains atleast one pivot element then \( RS(Y) \) is connected to \( (X_{m+1}), RS(X_{m+2}), \ldots RS(X_n) \). If \( Y \) does not contain any pivot element then \( RS(Y) \) is connected to any one of \( (x_1), RS(x_2), \ldots RS(x_m) \). Therefore \( D \) is the Dominating set.

Next let us prove that \( D \) is minimal. We know that removal of any one element in \( D \) will affect the domination property. If we remove \( RS(x_1) \) from \( D \) then let \( D_1 = D - RS(x_1) \) then \( RS(x_1) \cup x_2 \cup \ldots \cup x_m \cup X_{m+1} \cup \ldots X_n \in Z(T^*) \) and it is not connected to any element in \( D_1 \). Hence \( D_1 \) cannot be a dominating set. Similar argument is true if we remove any of \( RS(x_2), RS(x_3) \ldots RS(x_m) \).

Now if \( RS(X_{m+1}) \) is removed from \( D \) and \( D_2 = D - RS(X_{m+1}) \) then \( RS(x_1), RS(x_2) \ldots RS(x_m), RS(X_{m+2}), \ldots RS(X_n) \in Z(T^*) \) and it is not connected to any element in \( D_2 \). Hence \( D_2 \) cannot be a dominating set. Similar argument is true if we remove any of \( RS(X_{m+2}), RS(X_{m+3}) \ldots RS(X_n) \). Therefore \( D \) is the minimal dominating set.

Theorem

The dominating number of the rough zero divisor graph \( G(Z(T^*)) \) is equal to the number of equivalence classes induced by \( R \) on \( U \).

Proof

Already we have proved that \( D = RS(x_1), RS(x_2), \ldots RS(x_m), RS(X_{m+1}), RS(X_{m+2}), \ldots RS(X_n) \) is a minimal Dominating set. Now we have to prove that \( D \) is the smallest minimal dominating set. That is to prove that if there is another dominating set \( \bar{D} \) of \( n_1 \) elements where \( n_1 < n \) then \( \bar{D} \) cannot be a minimal dominating set.

On the contrary, if we assume that \( \bar{D} \) is a minimal dominating set consisting of \( n_1 \) elements where \( n_1 < n \). Let \( \bar{D} = \{ RS(Y_1), RS(Y_2), \ldots RS(Y_{n_1}) \} \). As there are \( n \) equivalence classes and \( \bar{D} \) contains \( n_1 \) elements where \( n_1 < n \), elements of \( \bar{D} \) cannot cover the representative of all the equivalence classes. Hence there exist atleast one equivalence class \( X_i \) such that either \( RS(X_i) \cup A \) or \( RS(x_i) \cup A \) does not belong to \( \bar{D} \). Where \( A \) is the union of none, one or more of equivalence classes or their representative elements other than \( X_i \).

Now \( A \) cannot contain the representative of all the remaining \( n - 1 \) equivalence classes (i.e excluding \( X_i \)) otherwise \( X_i \cup A \) becomes a dominating set. Hence there exist atleast one equivalence classes say \( X_j \) or its representative element \( x_j \). But in that case \( x_j \cup x_2 \cup \ldots x_{j-1} \cup x_{j+1} \cup \ldots x_m \cup X_{m+1} \cup \ldots X_n \) is not connected to any element in \( \bar{D} \). But it belongs to \( Z(T^*) \). Hence \( \bar{D} \) cannot be a dominating set which is a contradiction. Therefore \( D \) is the smallest minimal dominating set. Hence the domination number of \( G(Z(T^*)) \) is equal to \( n \) which is a number of equivalence classes induced by \( R \) on \( U \).

Lemma

The vertex \( RS(x_i) \) in \( G(Z(T^*)) \) is connected to \( \alpha_i = \{ RS(Y); Y = (Z_1 \cup Z_2 \ldots Z_{i-1} \cup Z_{i+1} \ldots Z_n) \neq \emptyset \} \) where \( Z_i \in B_i \) and for \( i = 1, 2, \ldots n \).

Proof

Note that elements of \( \alpha_i \) are of the form \( RS(Y) \) where \( Y \) does not contain \( X_i \) or its pivot element \( x_i \). Hence \( RS(x_i) \nabla RS(Y) = RS( \emptyset ) \). Hence \( RS(x_i) \) and \( RS(Y) \) are adjacent in \( G(Z(T^*)) \).

Lemma

The vertex \( RS(x_i) \) in \( G(Z(T^*)) \) is connected to \( \beta_i = \{ RS(Y); Y = (Z_1 \cup Z_2 \ldots Z_m \cup Z_{m+1} \ldots Z_{i-1} \cup Z_{i+1} \ldots Z_n) \neq \emptyset \} \) where \( Z_i \in B_i \) and for \( i = m + 1, m + 2, \ldots n \).

Proof

Can be proved in similar lines as in lemma 3.1

Lemma

(i) The degree of \( RS(x_i) \) in \( G(Z(T^*)) \) is \( 3^{m-1}2^{n-m} - 1 \) for \( i = 1, 2, \ldots n \).

(ii) The degree of \( RS(x_i) \) in \( G(Z(T^*)) \) is \( 3^{m-1}2^{n-m} - 1 \) for \( i = m + 1, m + 2, \ldots n \).

Proof

(i) First let us find degree of \( RS(x_1) \) by lemma 3.1 \( RS(x_1) \) is connected to \( \alpha_i = \{ RS(Y); Y = (Z_2 \cup Z_3 \ldots Z_n) \neq \emptyset \} \). Note that each \( Z_i, i = 1, 2, \ldots m \) have three possible elements. Hence
$Z_2 \cup Z_3 \cup \ldots \cup Z_m$ can be taken in $3^{m-1}$ ways. Also each $Z_i$ for $i = m + 1, m + 2, \ldots, n$ have two possible elements which can be taken from $B_i$ for $i = m + 1, m + 2, \ldots, n$. Hence $Z_{m+1} \cup Z_{m+2} \cup \ldots \cup Z_n$ will have $2^{n-m}$ possible elements. Hence $RS(Y)$ in $\alpha_i$ could be any of $3^{m-1}2^{n-m} - 1$ elements excluding $RS(\emptyset)$. Hence $RS(x_i)$ is connected to $x_i$, where $|\alpha_i| = 3^{m-1}2^{n-m} - 1$. Hence $|\alpha_i| = 3^{m-1}2^{n-m} - 1$ where $i = 1, 2, \ldots, m$.

Therefore degree of $RS(x_i)$, $i = 1, 2, \ldots, m$ is $3^{m-1}2^{n-m} - 1$.

(ii) The proof of this can be given in similar lines as above.

**ILLUSTRATION**

In this section we illustrate the above concept through example.

**Example**

Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$

Let $\{X_1, X_2, X_3\}$ are the equivalence classes induced by an equivalence relation $R$ on $U$. Where

$X_1 = \{x_1, x_3\}, X_2 = \{x_2, x_4, x_6\}, X_3 = \{x_5\}$

$T = \{RS(\emptyset), RS(X_1), RS(X_2), RS(X_3), RS(\{x_1\}), RS(\{x_2\}), RS(\{x_3\}), RS(\{x_1, x_2\}), RS(\{x_1, x_3\}), RS(\{x_2, x_3\}), RS(\{x_1, x_2, x_3\}), RS(\{x_1, x_2\})\}$

$Z(T') = \{RS(X_1), RS(X_2), RS(X_3), RS(\{x_1\}), RS(\{x_2\}), RS(\{x_3\}), RS(\{x_1, x_2\}), RS(\{x_1, x_3\}), RS(\{x_2, x_3\}), RS(\{x_1, x_2, x_3\}), RS(\{x_1, x_2\})\}$

Then the corresponding rough zero divisor graph $G(Z(T'))$

![Rough zero divisor graph](image)

**APPLICATION**

In the following discussion we illustrate the use of rough zero divisor graph in medicine. Let $U$ be the set of symptoms. Let $X_1$ be the disease Malaria ($B_1$) with symptoms fever, sweating, vomiting, diarrhea and $X_2$ be the disease Heart attack ($B_2$) with symptoms abdominal pain, nausea, shortness of breath, fatigue. Clearly the set $\{x_1\}$ and $\{x_2\}$ forms a partition on $U$. Let $x_1 = fever$ and $x_2 = abdominal pain$ be the representative of the classes.

$T = \{RS(\emptyset), RS(X_1), RS(X_2), RS(\{x_1\}), RS(\{x_2\}), RS(\{x_1, x_2\}), RS(\{x_1\} \cup X_2), RS(\{x_1, x_2\})\}$
If \( X = \{ \text{sweating, nausea} \} \) then \( RS(X) = \{ \emptyset, X, X \cup X \} \) this means that these two symptoms together cannot confirm and \( D_1 \) and \( D_2 \) but these two are the symptoms of either \( D_1 \) or \( D_2 \).

If \( X = \{ \text{fever, sweating, vomiting, diarrhea} \} \) then \( RS(X) = \{ X, X \} \) means that it conforms \( D_1 \). A similar interpretation can be given to all the rough sets in \( T \).

The rough zero divisors \( Z(T^*) = \{ RS(X_1), RS(X_2), RS(\{x_1\}), RS(\{x_2\}) \} \) and the rough zero divisor graph is

![Rough zero divisor graph](image)

**Figure 2: Rough zero divisor graph**

Now for any two subsets \( X \) and \( Y \) of \( U \). \( XY^c \) reflects the common symptoms with respect to the diseases \( D_1 \) and \( D_2 \). Hence if \( RS(X) \) is a rough zero divisor then among the symptoms in \( U \) we are able to identify a subset \( Y \) that does not have any common symptom with \( X \). In the rough zero divisor graph two such elements are connected. Therefore the edge \( RS(X) \) to \( RS(Y) \) in \( G(Z(T^*)) \) indicates that \( X \) and \( Y \) does not have any symptom in common. Also note that a Dominant set in this case represents a set of symptoms that exists with respect to both \( D_1 \) and \( D_2 \).

(i.e.), if \( X = \{ \text{fever, abdominal pain} \} \) is a Dominant set and \( RS(X) = (U, U) \) which means that it is necessary to check for both \( D_1 \) and \( D_2 \).

**CONCLUSION**

In this paper we have proved that \( D \) is a minimal dominating set of \( G(Z(T^*)) \) of rough semiring \( T \). Also we have proved that the domination number of \( G(Z(T^*)) \) is equal to the number equivalence classes \( R \) on \( U \), these concepts are illustrated through example. The future work in this direction is to explore the applications of domination number of rough zero divisor graph of rough semiring.

**REFERENCES**


