

# An analytical approach for coupled KdV and Modified Camassa-Holm equations

Mohd. Junaid Siddiqui<sup>1</sup>, Azhar Iqbal<sup>2</sup>

<sup>1,2</sup>Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University (PMU), 31952 Al Khobar, Kingdom of Saudi Arabia

Received 14 June 2017 • Revised 23 August 2017 • Accepted 24 September 2017

**Abstract:** *Using the Reduced Differential Transform Method (RDTM), it is possible to find the exact solutions or better approximate solutions of wide classes of problems in mathematical physics. The coupled KdV equations introduced by Hirota-Satsuma and Modified Camassa-Holm equation introduced by Wazwaz have rich applications in mathematical physics. In this paper, this method is used for solving these nonlinear equations with given initial conditions containing arbitrary constants. The numerical solutions obtained by RDTM are compared with the exact known solutions by fixing the arbitrary constants. The obtained results show that the solutions obtained by RDTM are in good agreement with the exact solutions.*

**Keywords:** *Reduced Differential Transform Method; Modified Camassa-Holm equation; Coupled Korteweg-de Vries (KdV) equations.*

## INTRODUCTION

There are many wave equations, which are quite useful and applicable in engineering and physics such as the well-known linear and nonlinear wave equations, the wave equation in an unbounded domain, kdv equation, nonlinear coupled Korteweg-de Vries (KdV) equations, Modified Camassa-Holm (MCH) equation and so on. Nonlinear wave phenomena plays a major role in sciences such as fluid mechanics, plasma physics, solid state physics, optical fibers, chemical kinetics and geochemistry. These problems include a vibrating string, vibrating membrane, shallow water waves, shock waves, chemical exchange processes in chromatography, sediment transport in rivers and waves in plasmas, and both electric and magnetic fields in the absence of charge and dielectric [1].

Various approximate and analytical methods have recently been developed to solve linear and nonlinear differential equations ([2-9]). One of the most important partial differential equations occurring in applied mathematics is Korteweg-de Vries (KdV) equation. Firstly Wadati [10] developed solutions of KdV and the mKdV equations ([11-12]). Here our interest is to obtain the solution of again an important nonlinear evolution equations in mathematical physics known as KdV coupled equations. The coupled KdV equations introduced by Hirota-Satsuma [13] describe interactions of two long waves with different dispersion relations.

The Camassa-Holm (CH) equation is a shallow water equation and was originally derived as an approximation to the incompressible Euler equation. One of the main features of the CH equation is that they admit "peakon" solutions. The name "peakon", which means travelling wave with slope discontinuities, is used to distinguish them from general travelling wave solutions since they have a corner at the peak of height  $c$ , where  $c$  is the wave speed. Since the CH equation has rich applications. Wazwaz [14] suggested a modified form of the MCH.

In this paper we will apply the Reduced differential transform method (RDTM) to solve the MCH equation and nonlinear coupled KdV equations. The main advantage of the method is the fact that it provides its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent series with elegantly computed terms.

## METHODOLOGY

The basic definitions in the reduced differential transform method [15] are as follows:

**Definition 2.1:** Let function  $u(x, t)$  be analytic and  $k$ -times continuously differentiable with respect to time  $t$  and space  $x$  in the domain of interest, and let

$$U_k(x) = \frac{1}{k!} \left( \frac{\partial^k u(x, t)}{\partial t^k} \right)_{t=0} \quad (1)$$

where the function  $U_k(x)$  is the transformed function of the original function  $u(x, t)$ . The differential inverse transform of  $U_k(x)$  is defined as

$$u(x, t) = \sum_{k=0}^{\infty} U_k(x) t^k \quad (2)$$

Then, combining equations (1) and (2), we write

$$u(x, t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{\partial^k u(x, t)}{\partial t^k} \right)_{t=0} t^k. \quad (3)$$

From the above definitions, it can be found that the concept of the RDT method is derived from the Taylor's series expansion.

Again, Definition 2.1 implies that the initial approximation  $U_0(x)$  is given by the initial condition, that is

$$U_0(x) = u(x, 0). \quad (4)$$

Taking the Reduced Differential Transformation of the equation to be solved, we obtain an iteration formula for  $U_k(x)$ . Then the differential inverse transformation of the set of values  $[U_k(x)]_{k=0}^n$  gives approximation solution as

$$\bar{u}_k(x, t) = \sum_{k=0}^n \frac{1}{k!} \left( \frac{\partial^k u(x, t)}{\partial t^k} \right)_{t=0} t^k. \quad (5)$$

Therefore, the differential inverse transform of  $U_k(x)$  is given by  $u(x, t) = \lim_{n \rightarrow \infty} \bar{u}_n(x, t)$ .

**Table 1:** Reduced Differential Transformation

Function	RDT
$u(x, t)$	$U_k(x) = \frac{1}{k!} \left( \frac{\partial^k u(x, t)}{\partial t^k} \right)_{t=0}$
$w(x, t) = u(x, t) \pm v(x, t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x, t) = \alpha u(x, t)$	$W_k(x) = \alpha U_k(x)$ ( $\alpha$ is constant)
$w(x, t) = \frac{\partial}{\partial x} u(x, t)$	$W_k(x) = \frac{\partial}{\partial x} (U_k(x))$
$w(x, t) = u(x, t)v(x, t)$	$W_k(x) = \sum_{r=0}^k V_r(x) U_{k-r}(x) = \sum_{r=0}^k U_r(x) V_{k-r}(x)$
$w(x, t) = \frac{\partial}{\partial t} u(x, t)$	$W_k(x) = (k+1)U_{k+1}(x)$

## APPLICATIONS

Two test problems are taken to show the efficiency and accuracy of the method. with the initial conditions using the RDTM. Throughout we have used  $\text{sech}(mx + \gamma t)^n$  for  $(\text{sech}(mx + \gamma t))^n$  and  $\tanh(mx + \gamma t)^n$  for  $(\tanh(mx + \gamma t))^n$ .

### 1. KdV coupled equations

**Table 2:** Comparison of RDTM solution with the exact solution of  $u(x, t)$  for  $\alpha = 1, \beta = 1, \lambda = 1$ .

t	x	RDTM	Exact	Absolute error
---	---	------	-------	----------------

0.1	1	0.822084	0.822001	0.000083
	0	0.997500	0.997504	0.000004
	-1	0.749398	0.749480	0.000082
0.2	1	0.856308	0.855639	0.000669
	0	0.990000	0.990066	0.000066
	-1	0.710935	0.711578	0.000643
0.3	1	0.889118	0.886851	0.002267
	0	0.977500	0.977833	0.000333
	-1	0.671060	0.673193	0.002133

In this subsection, we will solve using the RDTM the following (1+1)-dimensional nonlinear KdV coupled [12] equations in the form:

$$u_t - \alpha u_{xxx} - 6\alpha u u_x + 2\beta v v_x = 0, \tag{6}$$

$$v_t + u_{xxx} + 3u v_x \tag{7}$$

with initial conditions

$$u(x, 0) = \lambda^2 (\operatorname{sech}(\lambda x/2))^2 \tag{8}$$

and

$$v(x, 0) = \frac{\lambda^2}{2} \sqrt{\frac{6\alpha}{\beta}} (\operatorname{sech}(\lambda x/2))^2 + \frac{1}{\sqrt{6\alpha\beta}} \lambda^2 (\alpha + 1). \tag{9}$$

Here  $u = u(x, t), v = v(x, t)$  are the solutions of (6) and (7), and  $\alpha$  and  $\beta$  are the nonzero positive constants.

The exact solution for the above problem is given by

$$u(x, t) = \lambda^2 (\operatorname{sech}(\lambda(x - \lambda^2 t)/2))^2 \tag{10}$$

and

$$v(x, t) = \frac{\lambda^2}{2} \sqrt{\frac{6\alpha}{\beta}} (\operatorname{sech}(\lambda(x - \lambda^2 t)/2))^2 + \frac{1}{\sqrt{6\alpha\beta}} \lambda^2 (\alpha + 1). \tag{11}$$

The coupled KdV equations (6) and (7) are introduced by Hirota-Satsuma [12] and describe interactions of two long waves with different dispersion relations. In [12], the authors showed that for all values of  $\alpha$  and  $\beta$ , the system (6) and (7) possesses three conservation laws and a solitary wave solution. In [15], the authors have found the exact and numerical traveling wave solutions of this system using the decomposition method. Let us now solve the system (6) and (7) by the RDTM method.

Taking the Reduced Differential transformation of both sides of equations (6) and (7), we obtain the iterative scheme as follows:

$$(k + 1)U_{k+1}(x) = \alpha \frac{\partial^3}{\partial x^3} (U_k(x)) + N_k(U_k(x)) + M_k(V_k(x)) \tag{12} \text{ and}$$

$$(k + 1)V_{k+1}(x) = -\frac{\partial^3}{\partial x^3} (V_k(x)) + L_k(U_k(x), V_k(x)) \tag{13}$$

where  $N_k(U_k(x))$  is the reduced differential transformation of  $6\alpha u u_x, M_k(V_k(x))$  is the reduced differential transformation of  $-2\beta v v_x$  and  $L_k(U_k(x), V_k(x))$  is the reduced differential transformation of  $-3u v_x$ .

Using the initial conditions (8) and (9), we obtain

$$U_0(x) = \lambda^2 (\operatorname{sech}(\lambda x/2))^2,$$

$$V_0(x) = (\lambda^2/2) \sqrt{6\alpha/\beta} (\operatorname{sech}(\lambda x/2))^2 + \lambda^2 (\alpha + 1) / \sqrt{6\alpha\beta},$$

$$U_1(x) = -6\alpha \lambda^5 (\operatorname{sech}(\lambda x/2))^4 \tanh(\lambda x/2) +$$

$$\sqrt{6} \sqrt{\frac{\alpha}{\beta}} \beta \lambda^3 (\operatorname{sech}(\lambda x/2))^2 \left( \frac{(1+\alpha)\lambda^2}{\sqrt{6}\sqrt{\alpha\beta}} + \sqrt{\frac{3}{2}} \sqrt{\frac{\alpha}{\beta}} \lambda^2 (\operatorname{sech}(\lambda x/2))^2 \right) \tanh(\lambda x/2) + \alpha \lambda^2 (2\lambda^3 (\operatorname{sech}(\lambda x/2))^4 \tanh(\lambda x/2) - \lambda^3 (\operatorname{sech}(\lambda x/2))^2 (\tanh(\lambda x/2))^3).$$

$$V_1(x) = 3 \sqrt{\frac{3}{2}} \sqrt{\frac{\alpha}{\beta}} \lambda^5 (\operatorname{sech}(\lambda x/2))^4 \tanh(\lambda x/2) - \sqrt{\frac{3}{2}} \sqrt{\frac{\alpha}{\beta}} \lambda^2 (2\lambda^3 (\operatorname{sech}(\lambda x/2))^4 \tanh(\lambda x/2) - \lambda^3 (\operatorname{sech}(\lambda x/2))^2 (\tanh(\lambda x/2))^3)$$

and so on.

Then, the differential inverse transformation of the set of values  $[U_k(x)]_{k=0}^2$  gives the second order approximation solution as

$$\begin{aligned} \bar{u}_2(x, t) &= \sum_{k=0}^2 U_k(x) t^k \\ &= U_0(x) + U_1(x)t + U_2(x)t^2, \end{aligned}$$

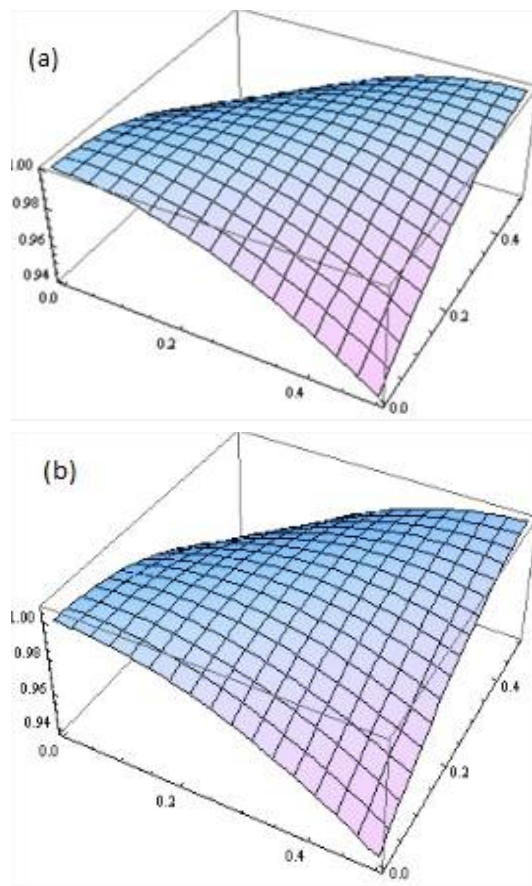
and the differential inverse transformation of the set of values  $[V_k(x)]_{k=0}^2$  gives the second order approximation solution as

$$\begin{aligned} \bar{v}_2(x, t) &= \sum_{k=0}^2 V_k(x) t^k \\ &= V_0(x) + V_1(x)t + V_2(x)t^2. \end{aligned}$$

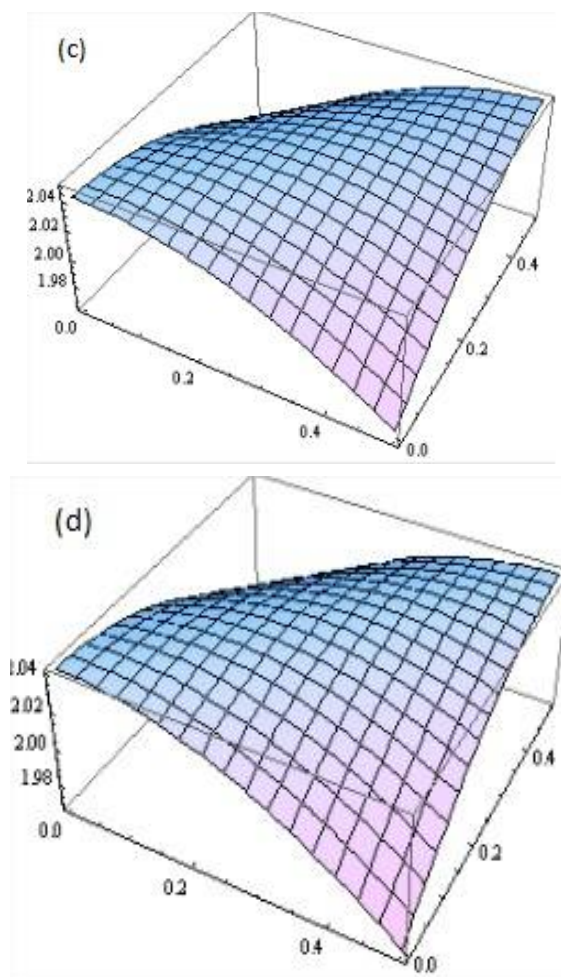
The comparison of the present approximate solution with the exact solution (10) of the coupled KdV equations is made in the following tables:

**Table 3:** Comparison of the approximate solution with the exact solution of  $v(x, t)$  for  $\alpha = 1, \beta = 1, \lambda = 1$ .

t	x	RDTM	Exact	Absolute error
0.1	1	1.82334	1.82324	0.0001
	0	2.03818	2.03818	0
	-1	1.73432	1.73442	0.0001
0.2	1	1.86524	1.86444	0.0008
	0	2.02899	2.02908	0.00009
	-1	1.68721	1.68800	0.00079
0.3	1	1.90544	1.90266	0.00278
	0	2.01368	2.01409	0.00041
	-1	1.63837	1.64099	0.00262



**Fig1:** The numerical results for  $\bar{u}_2(x, t)$  (given in (a)) is comparison with the exact analytical solutions  $u(x, t)$  (given in (b))



**Fig 2:** The numerical results for  $\bar{v}_2(x, t)$  (given in (c)) in comparison with the exact analytical solutions  $u(x, t)$  (given in (d)).

## 2. Modified Camassa-Holm equation (MCH) equation

We consider the modified Camassa-Holm equation (MCH) equation

$$u_t - u_{xxt} + 3u^2u_x = 2u_xu_{xx} + uu_{xxx}, \quad x \in \mathbb{R}, \quad t > 0, \quad (14)$$

with initial condition

$$u(x, 0) = k - \alpha (\operatorname{sech}(mx))^2. \quad (15)$$

Here  $u = u(x, t)$  is the solution of (14).

We know that the exact solution for above problem is

$$u(x, t) = k - \alpha (\operatorname{sech}(mx + \gamma t))^2. \quad (16)$$

$$\text{where } \alpha = 1 + 2k + \sqrt{1 - 2k - 2k^2}, \quad m = \frac{\sqrt{2}\alpha}{4} \text{ and } \gamma = \frac{\sqrt{2}\alpha}{4}(3k - \alpha).$$

Using RDTM, the transformed form of equation (14) is

$$(\mathbf{k} + 1)U_{k+1}(x) = N_k(U_k(x)) + \frac{\partial^3}{\partial x^2 \partial t} \bar{u}_k(x, t) \quad (17)$$

where  $N_k(U_k(x))$  is the Reduced Differential Transformation of  $2u_xu_{xx} + uu_{xxx} - 3u^2u_x$ .

Using the initial condition (15), we obtain

$$U_0(x) = k - \alpha (\operatorname{sech}(mx))^2, \\ U_1(x) = -6m\alpha (\operatorname{sech}(mx))^2 (k - \alpha (\operatorname{sech}(mx))^2)^2 (\tanh(mx))$$

$-4m\alpha^2(\operatorname{sech}(mx))^2(\tanh(mx))(-2m^2(\operatorname{sech}(mx))^4 + 4m^2(\operatorname{sech}(mx))^2(\tanh(mx))^2) -$   
 $\alpha(k - \alpha(\operatorname{sech}(mx))^2)(16m^3(\operatorname{sech}(mx))^4(\tanh(mx)) - 8m^3(\operatorname{sech}(mx))^2(\tanh(mx))^3),$   
 and so on.

Then, the differential inverse transformation of the set of values  $[U_k(x)]_{k=0}^2$  gives the second order approximation solution as

$$\begin{aligned} \bar{u}_2(x, t) &= \sum_{k=0}^2 U_k(x) t^k \\ &= U_0(x) + U_1(x)t + U_2(x)t^2. \end{aligned}$$

The comparison of the present approximate solutions with the exact solutions of the MCH equation is made in the following table:

**Table 4:** Comparison of the approximate solution with the exact solution of MCH equation for  $k = 0$ .

t	x	RDTM	Exact	Absolute error
0.0001	0.5	-1.57241	-1.57304	0.00063
	0.0	-1.99999	-2.00000	0.00001
	-0.5	-1.5734	-1.57275	0.00065
0.0003	0.5	-1.57149	-1.57333	0.00184
	0.0	-1.99987	-2.00000	0.00013
	-0.5	-1.57444	-1.57246	0.00198
0.0005	0.5	-1.57062	-1.57362	0.00300
	0.0	-1.99964	-2.00000	0.00036
	-0.5	-1.57554	-1.57217	0.00337

### CONCLUSION

The main aim of this article is to construct approximate analytical solutions of the nonlinear coupled Korteweg–de Vries equations and Modified Camassa-Holm equation. We have achieved this goal by applying the Reduced Differential transform method. Its rapid convergence shows that the method is reliable and introduces a significant improvement in solving the nonlinear coupled Korteweg–de Vries equations and Modified Camassa-Holm equation over existing numerical methods. As the method is usually tedious to use by hand, we have used the software package “MATHEMATICA” to calculate the terms of the series obtained from the RDTM. The numerical results are compared with the exact solutions in Tables 2, 3 and 4. The approximate solutions are also compared with the exact solutions in Figures 1 and 2.

## REFERENCES

- [1]. L. Debnath, *Nonlinear partial differential equations for scientists and engineers*, Birkhauser, Boston, 1997.
- [2]. S. A. El-Wakil, E. M. Abulwafa and M. A. Abdou, An Improved variational iteration method for solving coupled KdV and Boussinesq-like B(m,n) equations, *Chaos, Solitons Fractals*, 39(3), 1324-1334, 2007.
- [3]. A. M. Wazwaz, The variational iteration method for solving linear and nonlinear systems of PDEs. *Computers & Mathematics with Applications*, 54(7-8), 895-902 2007.
- [4]. A. M. Wazwaz, Comparison between the variational iteration method and Adomian decomposition method, *J. Comput. Appl. Math*, 207(1), 129-136, 2007.
- [5]. G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer, 1994.
- [6]. G. Adomian, A review of the decomposition method in applied mathematics. *J. Math. Anal. Appl.*, 135, 501-544, (1988).
- [7]. A. M. Wazwaz, A Sine-Cosine method for handling Nonlinear Wave Equations, *mathematical and Computer modelling*, 40, 499-508, 2004.
- [8]. J. H. He, Variational iteration method-a kind of non-linear analytical technique: Some examples, *International Journal of Non-Linear Mechanics*, 34(4), 699-708, 1999.
- [9]. J. H. He, Homotopy Perturbation method, A new nonlinear technique, *Applied Mathematics and computation*, 135, 73-79, 2003.
- [10]. Manisha Kotadiya, Olinirina Nisha Andriambololona. "Recent Advances in Tuberculosis Drug Development ." *International Journal of Pharmacy Research & Technology* 8.2 (2018), 01-13.
- [11]. Siddaiah, N., Roshini, T., Sai Krishna, V., Prasanth, G., Likhith, K. "Performance analysis of cantilever based bio-sensor for pathogen detection" (2018) *International Journal of Pharmaceutical Research*, 10 (2), pp. 107-109.
- [12]. M. Wadati, The exact solution of the modified Kortweg-de Vries equation, *J. Phys. Soc. Jpn.*, 32, 1681-1687, 1972.
- [13]. M. Wadati, The modified Kortweg-de Vries equation, *J. Phys. Soc. Jpn.*, 34, 1289-1296, 1973.
- [14]. R. Hirota and J. Satsuma, Solutions of a coupled Korteweg-de Vries equation. *Phys. Lett. A*, 85, 407-408, 1981.
- [15]. A. M. Wazwaz, Solitary wave solutions for modified forms of Degasperis-Procesi and Camassa-Holm equations, *Phys. Lett. A*, 352, 500-504, 2006.
- [16]. Y. Keskin and G. Oturanc, Reduced Differential Transform Method for Partial Differential equations, *International Journal of Nonlinear Sciences and Numerical Simulation*, 10(6), 741-749, 2009.